MP 732, Homework 1

Due Thursday Jan. 24, 2019 at 11:59 P.M. on Sakai

January 24, 2019

Problem 1, 12 pts total. Consider a one-dimensional continuous, real-valued signal $f(x)$ observed in the spatial domain, with Fourier transform $\text{FT}\{f(x)\} = \hat{f}(\omega)$. For simplicity, assume that we are working in a coordinate system where $\mu_x = \mu_\omega = 0$.

The definitions

$$\mu_x = \frac{\int x f(x) dx}{\int f(x) dx}, \quad \mu_\omega = \frac{\int \omega \hat{f}(\omega) d\omega}{\int \hat{f}(\omega) d\omega}$$

$$\sigma_x^2 = \frac{\int (x - \mu_x)^2 |f(x)|^2 \, dx}{\int |f(x)|^2 \, dx}, \quad \sigma_\omega^2 = \frac{\int (\omega - \mu_\omega)^2 \left|\hat{f}(\omega)\right|^2 d\omega}{\int \left|\hat{f}(\omega)\right|^2 d\omega}$$

and relations

$$\text{FT}\left\{\frac{d}{dx} f(x)\right\} = 2\pi i \omega \hat{f}(\omega)$$ (Hint: This will help you to write both σ_x and σ_ω together in a single coordinate system)

$$|\sigma_x \sigma_\omega| \leq |\sigma_x| \, |\sigma_\omega|$$ (Hint: These standard deviations *really, really* need to be in the same coordinate system)

$$\int |f(x)|^2 \, dx = \int \left|\hat{f}(\omega)\right|^2 d\omega \quad \text{(Parseval's theorem)}$$

might be helpful below.

Part A, 8 pts. Derive the expression that gives the absolute lower bound on the product of the spatial domain and frequency domain standard deviations, namely $\sigma_x \sigma_\omega \geq \frac{1}{4\pi}$, and hence demonstrate that ideal spatial-domain localization and ideal frequency-domain localization are mutually exclusive.

(Hint: If you end up at the expression

$$\frac{\left|\int x f(x) f'(x) dx\right|}{2\pi \int |f(x)|^2 \, dx} \leq |\sigma_x| \, |\sigma_\omega|$$

you're on the right track.)

Part B, 2 pts. Notice that the inequality in the hint above is minimized if $f' \propto xf(x)$. What does this imply about the optimal functional form of $f(x)$? Does this make sense given what we know about this function and its Fourier transform?

Part C, 2 pts. The Dirac delta gives ideal spatial-domain localization at the expense of frequency-domain localization. What is the frequency-domain analog of the Dirac delta? Can you think of a modality where ideal frequency-domain localization might be useful? How is the spatial information actually "preserved" such that we can still get anatomical images in this modality? (Hint: Think beyond just the ionizing modalities)

Problem 2, 8 pts total. Suppose a radiologist is reading a screening mammogram and is following up on a suspicious, but—over the course of several screening mammograms—stable mass with well-circumscribed margins. For clinical action to be taken, suppose the radiologist needs to see the mass change such that it has an apparent diameter of 8mm. Let's approximate how large the ground-truth mass has to become for the radiologist to definitively say that the mass is 8mm in diameter in the image.

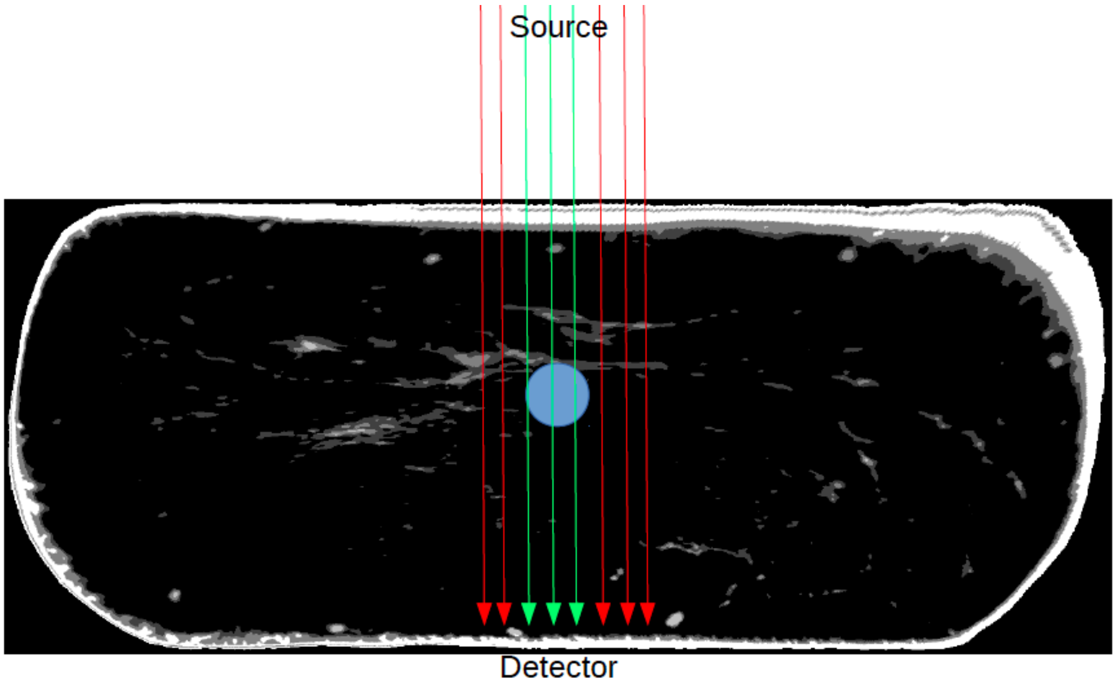

Figure 1: Coronal slice of the problem

Part A. Approximate the mass as a sphere of radius R with linear attenuation $\mu_{\text{mass}} = 0.9\text{cm}^{-1}$ surrounded locally by fibroglandular tissue with linear attenuation $\mu_{\text{fibroglandular}} = 0.8\text{cm}^{-1}$, in a compressed breast of thickness T. Assume the Rose model for minimum detectable contrast between an object and its background under high noise conditions, approximated here as

$$c_{\min} = \frac{\sqrt{3/\pi}}{100R}.$$

What is the ground-truth diameter of the mass when it can be confirmed by the radiologist to be 8mm in the image? Plot the solution region with R on one axis and T on the other with reasonable bounds. Include any code used.